# Generalized Solution of Freezing a Saturated Liquid in a Convex Container

**LUH C. TAO**

**University of Nebraska, Lincoln, Nebraska**

**Computer results of the required time to freeze completely a pure liquid in a container shaped like a slab, a cylinder or a sphere are correlated by equations which converge to the asymptotic solutions of the problem. Variables are grouped into dimensionless parameters so that these equations are applicable to any given system.**

The moving interface problem of freezing or thawing operation has many industrial applications. Several recent papers (*1* to *11*) have discussed numerical methods and solutions by digital and analog computers. This article extends the previous work (*9*) to include a generalized solution for a slab. Consequently, approximate solutions in the form of equations are obtained by correlating the computer results so that the time required to completely freeze the pure liquid inside a container can be estimated without repeating the numerical integration on a digital computer. All equations were so chosen as to converge to their respective asymptotic solutions.

The conventional containers are usually the slightly deformed shapes of either a slab, a cylinder, or a sphere. To accommodate the effect of small deformation of these three basic shapes, the use of an equivalent radius is suggested.

## SOLUTION OF FREEZING A SLAB

The physical operation of freezing a saturated liquid inside a slab container with a thickness of $2r_i$ can be described by the following set of equations for a system having a constant density and thermal coefficients (*2, 9*). Also, it is assumed that the solid-liquid interface is a smooth one.

$$\gamma \frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial r^{*2}} \quad \text{for} \quad R^* < r^* < 1 \tag{1}$$

$$\frac{dR^*}{dt^*} = \frac{\partial T^*}{\partial r^*} \quad \text{at} \quad r^* = R^* \tag{2}$$

$$-\frac{\partial T^*}{\partial r^*} = \frac{T_i^*}{\beta} \quad \text{at} \quad r^* = 1 \tag{3}$$

$$T^* = 1 \quad \text{at} \quad t^* = 0, \quad \text{for } 0 \leq r^* \leq 1 \tag{4}$$

where

$\beta = k/U_i r_i$ = 1/Nusselt number
$\gamma = C(T_L - T_C)/L$ = relative energy content
$T^* = (T - T_C)/(T_L - T_C)$ = dimensionless temperature
$t^* = tk(T_L - T_C)/r_i^2 \rho L$ = dimensionless time
$r^* = r/r_i$ = dimensionless position
$R^* = R/r_i$ = dimensionless interface position

Following the method described previously (*9*), numerical solutions are obtained by an IBM 360 computer with $\Delta r^* = 0.025$. The numerical solutions have been tabulated with $\gamma$ and $\beta$ up to 3 and 5 respectively.*

* The complete tabulation of results has been deposited as document #9978 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D.C., and are available for $1.25 for photograph or $1.25 for 35mm. microfilm.

An error analysis of similar computer results has been discussed (*9*). In addition, the present results may be examined by comparing with the two available asymptotic solutions. The asymptotic solution of $\gamma = 0$ is Equation (5) which can be derived similarly to those for a cylinder and a sphere (*9*). The other asymptotic (Neumann) solution of $\beta = 0$ is Equation (6) (*2*). The tabulated results show the convergence toward Equation (5) and Table 1 compares the computer solution and Equation (6). The largest error in Table 1 is about 0.5%:

$$t^* = (\beta + 1)(1 - R^*) - 0.5(1 - R^{*2}) \tag{5}$$

$$t^* = \gamma \Delta R^{*2}/4\lambda^2 \tag{6}$$

where $\lambda$ is the root of $\lambda \exp \lambda^2 \operatorname{erf} \lambda = \gamma/\sqrt{\pi}$

Table 2 compares further the present digital computer solution with the analog computer solution (*1*). The differences may be mainly due to that the analog computer solution used only five increments in comparison to forty increments in the present digital computer solution.

TABLE 1. A COMPARISON OF ASYMPTOTIC SOLUTIONS ($\beta = 0$)
(Slab container)

| | $\gamma = 0.1$ | 0.5 | 1.0 | 2.0 | 3.0 |
|---|---|---|---|---|---|
| Equation (6): $\gamma$ | 0.221 | 0.463 | 0.619 | 0.802 | 0.920 |
| $t^*(R^* = 0)$ | 0.0526 | 0.5831 | 0.6520 | 0.7780 | 0.8870 |
| Computer solution: $t^*(R^* = 0)$ | 0.0527 | 0.5861 | 0.6534 | 0.7764 | 0.8884 |

TABLE 2. A COMPARISON OF $t^*(R^* = 0)$ FROM DIFFERENT COMPUTERS
(Slab container)

| | | (reference *1*) | this work |
|---|---|---|---|
| $\beta = 1.0$ | $\gamma = 1.0$ | 1.84 | 1.801 |
| | 2.0 | 2.26 | 2.045 |
| $\beta = 0.5$ | $\gamma = 1.0$ | 1.30 | 1.251 |
| | 2.0 | 1.50 | 1.454 |
| $\beta = 0.1$ | $\gamma = 1.0$ | 0.76 | 0.782 |
| | 2.0 | 0.94 | 0.926 |

## CORRELATIONS OF COMPUTER SOLUTIONS

The use of the tabulated computer results for a specific problem requires interpolation. Therefore, attempts are made to correlate these generalized computer solutions to facilitate calculations in design work. These equations correlate the time required to completely freeze a slab, a cylinder, or a sphere. The functions are so chosen to force these equations to converge to their respective asymptotic solutions. Therefore, these correlations are indeed the approximate generalized solutions of the present freezing or thawing problem.

Equations (7), (8), and (9), are, respectively, correlations for a slab, cylinder, and sphere. All correlations converge to their respective asymptotic solutions of $\gamma = 0$. The averages of absolute errors and the maximum errors are respectively 0.7 and 2.6% for slabs, 0.7 and 2.4% for cylinders, and 0.9 and 2.6% for spheres, thus

$$t^*_{sl} = 0.5 + \beta + A_1 + B_1\beta - C_1 e^{-\beta} \quad (7)$$

$$A_1 = 0.0701\,[1 - \exp(-2.8\gamma)] + 0.27\gamma$$

$$B_1 = 0.0732\,[1 - \exp(-1.07\gamma)] + 0.137\gamma$$

$$C_1 = [0.016 - 0.0217 \exp(-1.72\gamma)]\gamma$$

$$t^*_{cy} = 0.25 + 0.5\beta + A_2 + B_2\beta - C_2 e^{-\beta} \quad (8)$$

$$A_2 = \gamma[0.25 + 0.23 \exp(-1.13\gamma)]$$

$$B_2 = \gamma[0.024 + 0.0512 \exp(-1.07\gamma)]$$

$$C_2 = \gamma[0.14 + 0.124 \exp(-0.95\gamma)]$$

$$t^*_{sp} = 1/6 + \beta/3 + A_3 + B_3\beta - C_3 e^{-\beta} \quad (9)$$

$$A_3 = 0.096\,[1 - \exp(5.49\gamma^3 - 6.62\gamma^2 - 4.59\gamma)] + 0.176\beta$$

$$B_3 = \gamma[0.024 + 0.179 \exp(-2.19\gamma)]$$

$$C_3 = 0.051\,[1 - \exp(-2.52\gamma)] + 0.104\gamma$$

## DISCUSSION

Equations (7) to (9) are interesting because they represent the approximate solutions of a set of coupled partial differential equations which cannot be solved analytically. The small errors of these correlations in comparison to the computer results suggest that they would be useful to the design work without repeating calculations for a specific ystem in a digital computer.

The slab, cylinder, and sphere are respectively symmetrical in one, two, and three dimensions. Prior to complete freezing, the final liquid region is a point, a line and a plane for a sphere, a cylinder, and a slab. Many industrially used containers are symmetrical shapes and thus may resemble one of the three basic shapes correlated. The choice of a reference shape is usually obvious and may also be guided by considering the geometric shape of the final liquid zone prior to complete freezing.

Any variation of container shape will change the time required to complete the freezing operation. Besides, to make numerical calculations by a digital computer for a specific shape, some limited extensions of Equations (7) to (9) may be made to cover many conventional containers which are essentially slightly deformed forms of those basic shapes. Such an extension may be made in several different ways depending on the specific shape given. Since the adjustments are to be made for slightly deformed shapes only, the particular basic shape and its equation or correlation are known. Adjustment may be made by using the given correlations with an equivalent radius defined by

$$r_i = (a)\,(\text{container volume/conductive surface area}) \quad (10)$$

where $a = 1, 2,$ or 3 for respectively a slab, a cylinder or a sphere.

An increase of conductive surface area of a given container volume would decrease $r_i$ and decrease the time required to complete the freezing operation.

The described correlations are also applicable to thawing a pure solid inside a container initially at the melting temperature provided that the frozen portion does not move during the thawing operation. For the thawing operation, $(T_L - T_C)$ and $(T - T_C)$ are to be replaced by $(T_C - T_L)$ and $(T_C - T)$ respectively.

## CONCLUSION

Equations are obtained as approximate solutions of a set of partial differential equations which can only be solved by using digital or analog computers. These equations containing dimensionless groups only are useful in calculating the time required to freeze or to thaw a pure liquid inside container which is a slightly deformed shape of a slab, cylinder, or a sphere. Constraints of these equations are (a) the container initially being at the melting temperature, (b) constant density, thermal diffusivity, and heat capacity, and (c) constant boundary condition at the outer surface of the container.

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## NOTATION

$a$ = constant in Equation (10)
$C$ = heat capacity
$k$ = thermal conductivity of solid
$L$ = latent heat of fusion
$r$ = radial distance of cylinder or sphere, or thickness parameter of a slab
$R$ = interface position
$t$ = time
$T$ = Temperature
$U$ = overall heat transfer coefficient
$\beta$ = 1/Nusselt number
$\Delta R^*$ = $1 - R^*$
$\gamma$ = relative energy content
$\lambda$ = constant in Equation (6)

### Superscript

$*$ = dimensionless variable

### Subscripts

$C$ = coolant
$i$ = based on inside wall
$L$ = at the melting point
cy = cylinder
sl = slab
sp = sphere

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